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On Faraday waves

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The standing waves of frequency ω and wavenumber k that are induced on the surface of a liquid of depth d that is subjected to the vertical displacement $a_0 \cos 2\omega t$ are determined on the assumptions that: the effects of lateral boundaries are negligible; $\epsilon = ka_0 \tanh kd \ll 1$ and $0 < \epsilon - \delta = O(\delta^3)$, where δ is the linear damping ratio of a free wave of frequency ω; the waves form a square pattern (which follows from observation). This problem, which goes back to Faraday (1831), has recently been treated by Ezerskii *et al.* (1986) and Milner (1991) in the limit of deep-water capillary waves ($kd, kl_* \gg 1$, where l_* is the capillary length). Ezerskii *et al.* show that the square pattern is unstable for sufficiently large $\epsilon - \delta$, and Milner shows that nonlinear damping is necessary for equilibration of the square pattern. The present formulation extends those of Ezerskii *et al.* and Milner to capillary–gravity waves and finite depth and incorporates third-order parametric forcing, which is neglected in these earlier formulations but is comparable with third-order damping. There are quantitative differences in the resulting evolution equations (for $kd, kl_* \gg 1$), which appear to reflect errors in the earlier work.

These formulations determine a locus of admissible waves, but they do not select a particular wave. The hypothesis that the selection process maximizes the energy-transfer rate to the Faraday wave selects the maximum of the resonance curve in a frequency-amplitude plane.

1. Introduction

I consider here standing waves in a container that is subjected to the vertical displacement

$$z = a_0 \cos 2\omega t \tag{1.1}$$

on the preliminary assumptions that

$$kb \gg 1, \quad 0 < \epsilon \equiv ka_0 \tanh kd \ll 1, \quad \delta = O(\epsilon), \quad \omega - \omega_k = O(\epsilon\omega), \qquad (1.2a\text{–}d)$$

where k is the wavenumber, b is the (minimum) breadth of the container, d is the depth, δ is the linear damping ratio of the wave,

$$\omega_k^2 = (gk + Tk^3) \tanh kd \equiv gk(1 + k^2 l_*^2) \tanh kd, \tag{1.3}$$

T is the kinematic surface tension, and l_* is the capillary length. The assumption $kb \gg 1$ permits the neglect of lateral boundaries.

This problem goes back to Faraday (1831), who discovered that the resulting waves (*Faraday waves*) have the frequency ω and a square pattern (the superposition of two standing waves of equal, orthogonal wavenumbers); see Miles & Henderson

(1990, hereinafter referred to as MH), for a review. Both observation and theory confirm that the free surface remains plane if ϵ is sufficiently small and that Faraday waves appear if ϵ exceeds a threshold for which theory yields $\epsilon = \delta$. (Comparison with the observed threshold is impeded by the uncertainty in δ, but agreement is typically within a factor of 2; see Milner 1991.) This prediction holds for any regular pattern (e.g. rolls, squares, or hexagons), the selection of which (for $kb \gg 1$) presumably depends on nonlinear effects. Faraday's observation of a square pattern has been confirmed by Rayleigh (1883), Ezerskii *et al.* (1986), Douady & Fauve (1988), and Tufillaro, Ramshankar & Gollub (1989) for sufficiently small supercriticality ($0 < \epsilon - \delta \ll 1$), and Milner (1991) offers theoretical support for this selection. The square pattern loses stability, and may become chaotic, for sufficiently large $\epsilon - \delta$.

The threshold $\epsilon = \delta$ represents a balance between linear parametric forcing and linear damping; however, as first recognized by Milner (1991), nonlinear (third-order) damping is necessary for a stationary pattern, which is achieved for $0 < \epsilon - \delta = O(\delta^3)$. This suggests that third-order parametric forcing, which Milner neglects, also may be significant.

The present investigation was stimulated by (what appear to be) errors or omissions in the formulations of Erzerskii *et al.* (1986) and Milner (1991). It is less comprehensive than either of these formulations in seeking only a description of Faraday waves for $0 < \epsilon - \delta = O(\delta^3)$ and eschewing a description of the instability of these waves for $\epsilon - \delta \gg \delta^3$, but it goes beyond them in treating finite depth and capillary–gravity waves (arbitrary kd and kl_* *vs.* $kd, kl_* \gg 1$ in their analyses) and in incorporating third-order forcing. Ezerskii *et al.* posit, without derivation ('we... make use of [Zakharov's] Hamiltonian description of the nonlinear interaction of capillary–gravity waves'), a set of evolution equations for the slowly varying (in both space and time), complex amplitudes of four plane waves of equal wavenumber for $kd, kl_* \gg 1$. These equations differ quantitatively from those of the present formulation in the special case of a square pattern (see Appendix C). Milner (1991) obtains evolution equations for the slowly varying amplitudes of an arbitrary set of plane waves for $kd, kl_* \gg 1$, but his results for a square pattern also differ quantitatively from those of the present formulation (see Appendix D).

Against this background, I proceed as follows. In §2, I pose a normal-mode expansion of the free-surface displacement that comprises the dominant mode $\psi_1(\boldsymbol{x})$, which describes the square pattern, and those secondary modes $\psi_n(\boldsymbol{x})$ for which $\langle \psi_1^2 \psi_n \rangle \neq 0$ ($\langle\ \rangle$ denotes a spatial average). In §3, I use these modes as the basis of a Lagrangian formulation, in which the amplitudes of the ψ_n are slowly varying sinusoids with carrier frequency ω for the dominant mode and 2ω for the secondary modes. The elimination of the secondary amplitudes and averaging over the period $2\pi/\omega$ then yields an average Lagrangian for the quadrature amplitudes of the envelope of the square pattern. In §4, I construct an average dissipation function that incorporates linear and third-order damping, and in §5 I combine the results of §§3 and 4 to obtain the evolution equations for the envelope. I then show that stationary states (corresponding to the stable fixed points of the evolution equations) other than the plane surface can exist only if $0 < \epsilon - \delta = O(\delta^3)$, introduce action-angle variables, and eliminate the angle to obtain a Landau equation for the action. (This reduction is equivalent to a centre-manifold projection.) In §6, I consider the resonance curve (the locus of stable fixed points) of this action equation in a frequency–energy plane and establish its bifurcation structure. It then remains to fix the location of the Faraday wave on this resonance curve.

In §4, I show that limit cycles are impossible in the two-dimensional phase plane of the envelope and hence, from the Poincaré–Bendixson theorem, that every solution of the evolution equations must terminate on a stable fixed point;† however, the present formulation does not select a particular fixed point and therefore leaves the amplitude of the Faraday wave undetermined. Milner (1991) constructs a Liapunov functional that vanishes for the null solution and assumes that the amplitude of the Faraday wave is selected by the requirement that this functional have the deepest possible minimum, but, in my view, his Liapunov functional is improper for the Faraday waves (see Appendix E). A more plausible hypothesis, it appears to me, is that the selection process maximizes the energy transfer to the Faraday wave, which criterion selects the maximum of the resonance curve. The corresponding r.m.s. (averaged over both space and time), free-surface displacement is given by

$$\langle \eta^2 \rangle^{\frac{1}{2}} = \left[2\left(\frac{\epsilon-\delta}{\gamma-P}\right)\right]^{\frac{1}{2}} k^{-1} \tanh kd, \tag{1.4}$$

where γ and P are measures of third-order damping and forcing, respectively, and are $O(\epsilon)$.

Following Milner (1991), I calculate γ (Appendix A) on the assumption that boundary-layer damping is negligible. This requires $kd \gg 1$ and an uncontaminated free surface, as in the experiments of Tufillaro *et al.* (1989); otherwise, γ may have to be determined experimentally.

2. Normal modes

Following MH, §2.1, we pose the free-surface displacement in the reference frame of the moving container in the form

$$\eta(\boldsymbol{x}, t) = \eta_n(t)\, \psi_n(\boldsymbol{x}; k_n), \tag{2.1}$$

where the ψ_n constitute a complete set of normal modes, k_n are the corresponding wave numbers, η_n are the corresponding generalized coordinates, and repeated dummy indices are summed over the participating modes except as noted. The participating ψ_n are determined by

$$(\nabla^2 + k^2)\,\psi = 0, \tag{2.2}$$

$$\langle \psi_m \psi_n \rangle = \delta_{mn}, \quad C_{11n} \equiv \langle \psi_1^2 \psi_n \rangle \neq 0, \tag{2.3a, b}$$

where δ_{mn} is the Kronecker delta, $\langle\ \rangle$ signifies an average over $\boldsymbol{x}$, C_{11n} measures the coupling between the primary mode ψ_1 and the secondary mode ψ_n, and $\psi_0 \equiv 1$ is excluded from (2.1) by conservation of mass.

The square pattern is described by the primary mode

$$\psi_1 = \cos kx + \cos ky \quad (k_1 = k). \tag{2.4}$$

The corresponding secondary modes, selected by (2.3*b*), are (we choose $n \equiv k_n^2/k^2$)

$$\psi_2 = 2\cos kx \cos ky \quad (k_2 = \sqrt{2}k), \qquad \psi_4 = \cos 2kx + \cos 2ky \quad (k_4 = 2k). \tag{2.5a, b}$$

† The instability of the regular pattern for sufficiently large $\epsilon - \delta$ is a consequence of spatial modulation, which is suppressed in the present formulation.

3 Average Lagrangian

The quartic truncation of the Lagrangian for the motion described by (2.1) is (MH, §2)

$$L = \tfrac{1}{2}(\delta_{mn}\, a_n + a_{lmn}\,\eta_l + \tfrac{1}{2}a_{jlmn}\,\eta_j\,\eta_l)\,\dot\eta_m\,\dot\eta_n - \tfrac{1}{2}[\delta_{mn}(g+\ddot z_0+Tk_n^2) - \tfrac{1}{4}T\ell_{jlmn}\,\eta_j\,\eta_l]\,\eta_m\,\eta_n, \quad (3.1)$$

where the fluid density has been factored out, g and $\ddot z_0$ are the gravitational and imposed accelerations, T is the kinematic surface tension,

$$a_n = (k_n \tanh k_n d)^{-1}, \quad (3.2a)$$

$$a_{lmn} = a_{lnm} = C_{lmn}[1+\tfrac{1}{2}(k_l^2-k_m^2-k_n^2)\,a_m\,a_n], \quad (3.2b)$$

$$a_{1111} = \tfrac{1}{2}C_{11n}^2\,a_1^2\,a_n\,k_n^4 - 2a_1\langle\psi_1^2(\nabla\psi_1)^2\rangle, \quad \ell_{1111} = \langle(\nabla\psi_1)^4\rangle, \quad (3.2c, d)$$

C_{11n} is defined by (2.3b), and (3.2c, d) anticipate that the quartic terms in (3.1) are significant in the present approximation only for $j = l = m = n = 1$.

Proceeding as in Miles (1984) and MH, §3, we pose the slowly varying amplitude of the primary mode in the form

$$\eta_1 = l[p(\tau)\cos\omega t + q(\tau)\sin\omega t], \quad \tau = \epsilon\omega t, \quad (3.3a, b)$$

where

$$l \equiv 2\epsilon^{\frac{1}{2}}k^{-1}\tanh kd, \quad \epsilon \equiv ka_0\tanh kd. \quad (3.4a, b)$$

Anticipating that $\eta_n = O(\epsilon)$ for the secondary modes and invoking Hamilton's principle, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot\eta_n}\right) - \frac{\partial L}{\partial\eta_n} = a_n(\ddot\eta_n+\omega_n^2\,\eta_n) + a_{11n}\frac{\mathrm{d}}{\mathrm{d}t}(\eta_1\,\dot\eta_1) - \tfrac{1}{2}a_{n11}\,\dot\eta_1^2 = 0, \quad (3.5)$$

where

$$\omega_n^2 = (gk_n + Tk_n^3)\tanh k_n d \equiv (g/a_n)(1+k_n^2\,l_*^2) \quad (3.6)$$

is the square of the natural frequency of the nth mode, and, here and subsequently, an error factor of $1+O(\epsilon)$ is implicit; n is not summed in (3.5), (3.6) and the remaining equations in this section. The required solution of (3.5) for η_n, regarded as forced by η_1, (3.3), is given by

$$\eta_n = (l^2/a_1)(A_n\cos 2\omega t + B_n\sin 2\omega t + C_n) \quad (n > 1), \quad (3.7)$$

where

$$(A_n, B_n) = \left(\frac{a_{n11}-4a_{11n}}{4\Omega_n}\right)(p^2-q^2, 2pq), \quad C_n = \frac{a_{n11}}{4\kappa_n}(p^2+q^2), \quad (3.8a, b)$$

$$\Omega_n \equiv \frac{4k\tanh kd}{k_n\tanh k_n d} - \kappa_n, \quad \kappa_n \equiv \frac{1+n\ell^2}{1+\ell^2}, \quad \ell \equiv kl_*. \quad (3.9a\text{–}c)$$

The hypothesis that the primary mode dominates the secondary modes fails in the neighbourhood of $\Omega_4 = 0$ owing to the resonance between modes 1 and 4 ($k_4 = 2k_1$ and $\omega_4 = 2\omega_1$, corresponding to Wilton's ripples), which we exclude. The denominator Ω_2 is positive-definite, whence resonance between modes 1 and 2 is impossible.

Substituting z_0, η_1 and η_n from (1.1), (3.3) and (3.7) into (3.1), averaging L over a 2π interval of ωt, invoking (3.4), and evaluating the modal coefficients $a_n, \ldots$, (3.2), for the ψ_n of (2.4) and (2.5), we obtain the average Lagrangian in the form

$$\langle L\rangle = a_0\,l^2\omega^2[\tfrac{1}{2}(\dot p q - p\dot q) + H(p,q)], \quad (3.10)$$

where the dots now imply differentiation with respect to the slow time τ,

$$H = \tfrac{1}{2}(p^2-q^2) + \tfrac{1}{2}P(p^4-q^4) + \tfrac{1}{2}\beta(p^2+q^2) + \tfrac{1}{4}C(p^2+q^2)^2 \quad (3.11)$$

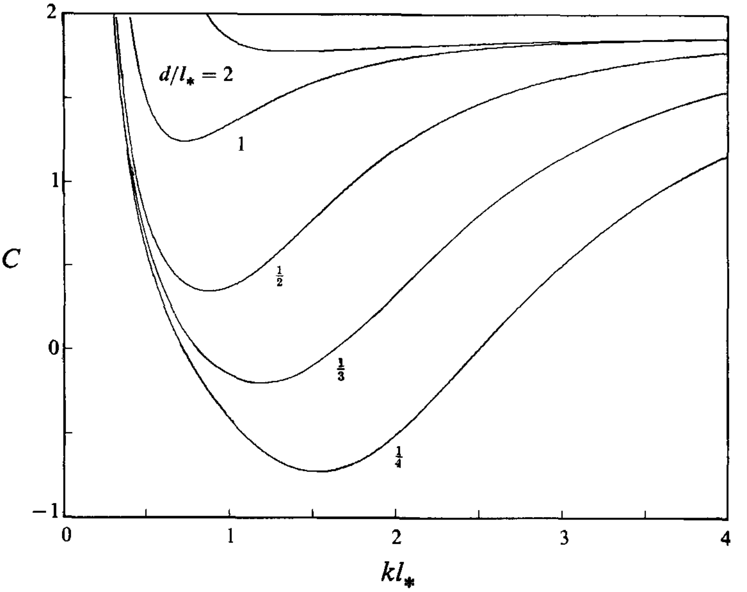

FIGURE 1. The scaling parameter C (3.13) for $d/l_* = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1 and 2. The steep rise of C on the left announces internal resonance between the primary wave and its second harmonic (Wilton's ripples). The asymptotic ($kl_* \uparrow \infty$) value of C is $(33-2\sqrt{2})/16 = 1.89$.

is a Hamiltonian for the slowly varying amplitudes,

$$\beta = (\omega^2 - \omega_1^2)/2\epsilon\omega^2 \tag{3.12}$$

is a measure of the frequency offset from (linear) resonance,

$$C = \tfrac{1}{4} + \tfrac{1}{2}S - \tfrac{1}{2}T^2 + \tfrac{15}{16}\sigma T^2 + \frac{1}{2}\frac{T^4}{\kappa_2} + \frac{(1+T^2)^2}{8\kappa_4} - \frac{(2S-3T^2)^2}{4\Omega_2} - \frac{(3-T^2)^2}{16\Omega_4} \tag{3.13}$$

(see figure 1) is a measure of the nonlinear interactions (Miles 1992),

$$\frac{P}{\epsilon} = \frac{(2S-3T^2)\,T^2}{\kappa_2\,\Omega_2} + \frac{(3-T^2)(1+T^2)}{4\kappa_4\,\Omega_4} \tag{3.14}$$

(see figure 2) is a measure of the parametric excitation of the secondary modes, and

$$S \equiv \frac{\sqrt{2}\tanh kd}{\tanh\sqrt{2}kd}, \quad T \equiv \tanh kd, \quad \sigma \equiv \frac{\ell^2}{1+\ell^2}. \tag{3.15a–c}$$

The retention of $\frac{1}{2}P(p^4-q^4)$ in H is, at first sight, inconsistent with the implicit neglect of other $O(\epsilon)$ terms therein, e.g. higher-order inertial terms; however, it proves to be directly comparable with nonlinear damping in the neighbourhood of the Faraday-wave threshold (see §5). We remark that $\langle L\rangle$ and H have the forms (3.10) and (3.11) for any regular pattern (e.g. rolls, squares, or hexagons), with C and P depending on the particular pattern. Rolls are considered in Appendix B.

The evolution equations implied by the requirement (Hamilton's principle) that $\langle L\rangle$ be stationary with respect to independent variations of p and q have the canonical form

$$\dot{p} = -\partial H/\partial q, \quad \dot{q} = \partial H/\partial p. \tag{3.16a, b}$$

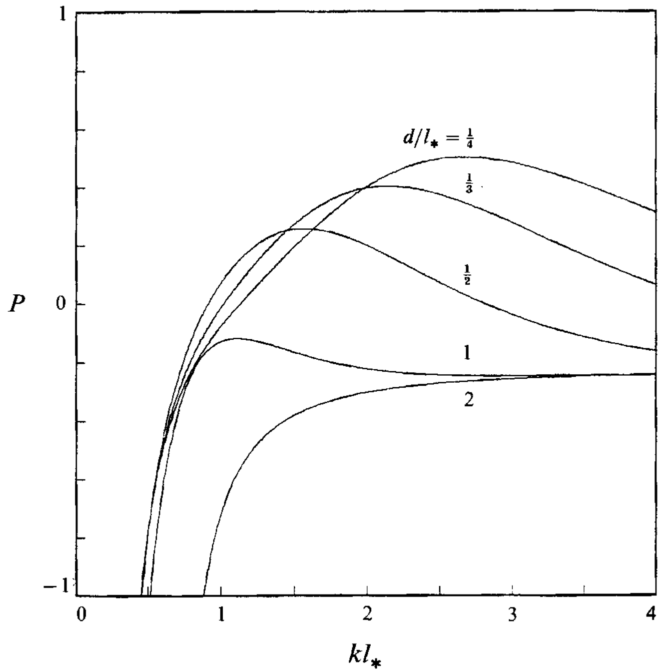

FIGURE 2. The parameter P (3.14) for $d/l_* = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1 and 2. The asymptotic ($kl_* \uparrow \infty$) value of P is $\frac{1}{8} - \frac{1}{4}\sqrt{2} = -0.23$.

4. Dissipation

We now posit a dissipation function $D(p, q)$ such that (3.16a, b) are replaced by

$$\dot{p} = -\frac{\partial D}{\partial p} - \frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial D}{\partial q} + \frac{\partial H}{\partial p}. \tag{4.1 a, b}$$

It follows from symmetry that D must have the form (cf. (3.11))

$$D = \tfrac{1}{2}\alpha(p^2+q^2) + \tfrac{1}{4}\gamma(p^2+q^2)^2, \tag{4.2}$$

where (by hypothesis) α, $\gamma > 0$, $\alpha = O(\delta/\epsilon)$, $\gamma = O(\delta)$, and δ is the linear damping ratio. The logarithmic contraction ratio for a closed orbit in the p, q phase plane is given by

$$\frac{\partial \dot{p}}{\partial p} + \frac{\partial \dot{q}}{\partial q} = -\left(\frac{\partial^2 D}{\partial p^2} + \frac{\partial^2 D}{\partial q^2}\right) = -2\alpha - 4\gamma(p^2+q^2) < 0, \tag{4.3}$$

in consequence of which limit cycles are impossible. It then follows from the Poincaré–Bendixson theorem that every solution of (4.1) must terminate on a stable fixed point.

We relate the parameters α and γ (which are independent of the excitation) to those of the dissipation function for the free motion through the mean energy equation,

$$(\mathrm{d}/\mathrm{d}t)\langle E\rangle = -2\langle F\rangle, \tag{4.4}$$

in which the energy E and the Rayleigh dissipation function F may be calculated (in the present approximation) for the undamped motion described by (2.1), (3.3) and (3.7). Proceeding as in §3 (in particular, the fluid density is factored out), we obtain (cf. (3.1) and (3.11))

$$\langle E\rangle = \tfrac{1}{2}\langle(\delta_{mn}\,a_n + a_{lmn}\,\eta_l + \tfrac{1}{2}a_{jlmn}\,\eta_j\,\eta_l)\,\dot\eta_m\,\dot\eta_n + [\delta_{mn}(g+Tk_n^2) - \tfrac{1}{4}T\mathscr{E}_{jlmn}\,\eta_j\,\eta_l]\,\eta_m\,\eta_n\rangle \tag{4.5a}$$

$$= a_0\,l^2\omega^2\left[\frac{1}{2}\left(\frac{\omega^2+\omega_1^2}{2\epsilon\omega^2}\right)(p^2+q^2) + \tfrac{1}{4}\hat{C}(p^2+q^2)^2\right], \tag{4.5b}$$

where $\hat{C}$ is given by (3.13) with the sign of σ therein reversed. The mean dissipation function may be developed in the corresponding forms (see, e.g. Appendix A)

$$\langle F\rangle = \langle(\delta_{mn} f_n + f_{lmn}\,\eta_l + f_{jlmn}\,\eta_j\,\eta_l)\,\dot\eta_m\,\dot\eta_n\rangle \tag{4.6a}$$

$$= \tfrac{1}{2}a_0\,l^2\omega^3[(\delta/\epsilon)(p^2+q^2) + \Gamma(p^2+q^2)^2], \tag{4.6b}$$

in which $\Gamma = O(\delta)$. Substituting (4.5b) and (4.6b) into (4.4), invoking $\tau = \epsilon\omega t$, $\omega_1^2 = \omega^2[1+O(\epsilon)]$, and $\Gamma, \delta = O(\epsilon)$, and expanding in powers of p^2+q^2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{p^2+q^2}{2}\right) = -(\delta/\epsilon)(p^2+q^2) - (\Gamma - \delta\hat{C})(p^2+q^2)^2. \tag{4.7}$$

The corresponding result implied by (4.1), (4.2) and the Hamiltonian for the free motion (cf. (3.11)),

$$H = \tfrac{1}{2}\beta(p^2+q^2) + \tfrac{1}{4}C(p^2+q^2)^2, \tag{4.8}$$

is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{p^2+q^2}{2}\right) = -\alpha(p^2+q^2) - \gamma(p^2+q^2)^2, \tag{4.9}$$

which may be compared with (4.7) to obtain

$$\alpha = \delta/\epsilon, \quad \gamma = \Gamma - \delta\hat{C}. \tag{4.10a, b}$$

It then remains to determine δ and Γ through calculation, as in Appendix A, or experiment.

Viscosity also may alter the resonant frequency. In particular, if δ is derived entirely from Stokes-like boundary layers at the bottom and free surface, ω_1 is reduced by a factor of $1-\delta$ from the value given by (3.6).

5. Action-angle reduction

Substituting (3.11) and (4.2) into (4.1), we obtain the evolution equations

$$\dot p = -[\alpha + \gamma(p^2+q^2)]\,p + (1+2Pq^2)\,q - [\beta + C(p^2+q^2)]\,q \tag{5.1a}$$

and

$$\dot q = -[\alpha + \gamma(p^2+q^2)]\,q + (1+2Pp^2)\,p + [\beta + C(p^2+q^2)]\,p. \tag{5.1b}$$

The fixed points of (5.1) correspond to either the null solution $p = q = 0$, which is stable (with respect to small disturbances) if $\alpha > 1$ and stable/unstable for $|\beta| \gtrless (1-\alpha^2)^{\frac{1}{2}}$ if $0 < \alpha < 1$, or Faraday waves with the threshold $\alpha = 1$ and $\beta = 0$ and a maximum at

$$p^2+q^2 = -\frac{\beta}{C} = \frac{1-\alpha}{\gamma - P} \quad (\alpha < 1). \tag{5.2}$$

Invoking (5.2) and $\gamma, P = O(\delta)$ in (5.1), we find that the remaining terms, which represent damping and parametric forcing, can balance (to yield $\dot p = \dot q = 0$) if and only if $1-\alpha = O(\delta^2)$.

Guided by these considerations, we rescale and introduce the action-angle variables (which are canonical variables for the rescaled Hamiltonian) $\mathscr{A}(\hat\tau)$ and $\theta(\hat\tau)$ according to

$$\begin{matrix} p \\ q \end{matrix} = \left(\frac{1-\alpha}{\gamma-P}\right)^{\frac{1}{2}} \mathscr{A}^{\frac{1}{2}} \begin{matrix}\cos\\ \sin\end{matrix}\,(\theta\,\mathrm{sgn}\,C + \tfrac{1}{4}\pi), \quad \hat\tau = (\epsilon - \delta)\,\omega t, \tag{5.3a, b}$$

in (5.1) to obtain

$$\dot{\mathscr{A}} = 2\mathscr{A}\left[1-\mathscr{A}-\left(\frac{2}{1-\alpha}\right)\sin^2\theta\right], \quad (1-\alpha)\,\dot{\theta} = \mu\left(\frac{1-\alpha}{2}\right)^{\frac{1}{2}}\mathscr{A}+(1-\alpha^2)^{\frac{1}{2}}B-\sin 2\theta, \tag{5.4a, b}$$

where
$$B \equiv \left(\frac{\omega-\omega_1}{\omega}\right)\frac{\operatorname{sgn} C}{(\epsilon^2-\delta^2)^{\frac{1}{2}}}, \quad \mu \equiv \frac{|C|}{\gamma-P}\left[\frac{2(\epsilon-\delta)}{\epsilon}\right]^{\frac{1}{2}}, \tag{5.5a, b}$$

the dots now signify $\mathrm{d}/\mathrm{d}\hat{\tau}$, and, here and subsequently, an error factor of $1+O(1-\alpha)$ is implicit (this error factor may be dominated by that of $1+O(\epsilon)$ already implicit in the truncation of the Lagrangian in §3). It follows from (5.4b) that θ may be approximated by

$$\theta = (\tfrac{1}{2}(1-\alpha))^{\frac{1}{2}}(B+\tfrac{1}{2}\mu\mathscr{A}), \tag{5.6}$$

the substitution of which into (5.4a) yields the Landau equation

$$\dot{\mathscr{A}} = 2\mathscr{A}[1-B^2-(1+\mu B)\mathscr{A}-\tfrac{1}{4}\mu^2\mathscr{A}^2] \equiv \dot{\mathscr{A}}(\mathscr{A};B,\mu). \tag{5.7}$$

6. The resonance curve

The fixed points of (5.7) are given by $\mathscr{A}=0$ and

$$\mathscr{A} = \frac{2(1-B^2)}{1+\mu B\pm(1+2\mu B+\mu^2)^{\frac{1}{2}}} \equiv \mathscr{A}_{\pm}(B,\mu). \tag{6.1}$$

If, as we henceforth assume, $\epsilon > \delta$ and $\gamma > P$† ($\mu > 0$) $\mathscr{A}=0$ is stable/unstable for $B^2 \gtrless 1$, while the upper/lower ($\mathscr{A}=\mathscr{A}_{\pm}$) branch of the resonance curve (6.1) in a $(B,\mathscr{A})$-plane is stable/unstable (i.e. is a locus of stable/unstable fixed points). The upper branch joins the null solution at a supercritical pitchfork bifurcation at $B=1$, and has a maximum at $B=-\frac{1}{2}\mu$ and $\mathscr{A}=1$ and joins the lower branch at a turning point (saddle-node bifurcation) at

$$B = -\tfrac{1}{2}(\mu+\mu^{-1}) \equiv B_*, \quad \mathscr{A} = 1-\mu^{-2}, \tag{6.2a, b}$$

if and only if $\mu > 1$. The lower branch then joins the null solution at a subcritical pitchfork bifurcation at $B=-1$. If $\mu < 1$ the lower branch disappears (into $\mathscr{A} < 0$), and the upper branch extends to the bifurcation at $B=-1$, which then is supercritical. The limit $\mu \downarrow 0$ yields the parabola

$$\mathscr{A}_+ = 1-B^2+O(\mu). \tag{6.3}$$

The asymptotic ($\hat{\tau}\uparrow\infty$) value of B must lie in $(B_*, 1)$ if $\mu > 1$ or $(-1,1)$ if $\mu < 1$ and presumably is selected through a physical process (see e.g. Manneville 1990, chs 9, 10) that is not described by the present formulation. Perhaps the most plausible conjecture is that this process maximizes the absolute energy-transfer rate $|\dot{\mathscr{A}}|$. Setting $\partial\dot{\mathscr{A}}/\partial B = 0$ with $\mathscr{A} \neq 0$), we obtain

$$B = -\tfrac{1}{2}\mu\mathscr{A}, \quad \dot{\mathscr{A}} = 2\mathscr{A}(1-\mathscr{A}), \tag{6.4a, b}$$

which intersects $\mathscr{A}=\mathscr{A}_+$ at the maximum. The corresponding r.m.s. value of the free-surface displacement is given by (1.4).

† It follows from (A 7) and (A 8) that $\gamma > P$ for $\epsilon-\delta = O(\delta^3)$ if $kd \gg 1$, but I have not proved that this is so for all kd. Fifth-order (in amplitude) terms may have to be included in the Landau equation if $\gamma-P$ is small.

Milner (1991) assumes that B is selected by the requirement that the 'Lyapunov functional' (cf. Appendix E)

$$V(A,B) \equiv -\int_0^A \dot{A}(A,B)\,\mathrm{d}A \tag{6.5}$$

of the amplitude $A \equiv \mathscr{A}^{\frac{1}{2}}$ (in the present notation) have the deepest possible minimum. Substituting

$$\dot{A} = A[1-B^2-(1+\mu B)A^2-\tfrac{1}{4}\mu^2A^4] \tag{6.6}$$

into (6.5) and invoking $\partial V/\partial B = 0$ and $A = \mathscr{A}^{\frac{1}{2}}$, we obtain

$$B = -\tfrac{1}{4}\mu\mathscr{A}, \quad \mathscr{A} = \frac{2}{1+(1+\frac{1}{4}\mu^2)^{\frac{1}{2}}}. \tag{6.7a, b}$$

This lies to the right of the maximum; however, it coincides with the maximum of the parabola (6.3) if $\mu \ll 1$, and yields $0.94 < \mathscr{A} < 1$ for $1 > \mu > 0$.

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Appendix A. Dissipation function ($kd \gg 1$)

If $k(\nu/\omega)^{\frac{1}{2}} \ll 1$ and boundary-layer damping is neglected (which requires $kd \gg 1$ and an uncontaminated free surface†) the rotational component of the flow is uniformly small compared with the irrotational component. The dissipation function then may be approximated by (Lamb 1932, §329(7), divided by ρS)

$$F = \nu S^{-1} \iint \mathrm{d}S \int_{-\infty}^{\eta} (\phi_{xx}^2+\phi_{yy}^2+\phi_{zz}^2+2\phi_{xy}^2+2\phi_{yz}^2+2\phi_{xz}^2)\,\mathrm{d}z, \tag{A 1}$$

where ν is the kinematic viscosity, ϕ is the velocity potential, and $z=0$ at the ambient free surface. Invoking $\nabla^2\phi = 0$ and $kd \gg 1$ (so that $a_n = 1/k_n$), we pose

$$\phi = \phi_n(t)\,\psi_n(\boldsymbol{x})\,\mathrm{e}^{k_n z} \tag{A 2}$$

as the complement of (2.1). The ϕ_n may be calculated as in Miles (1976, §2) and are given by

$$\phi_n = \delta_{mn} k_m^{-1}\dot{\eta}_m + \left[\frac{k_l^2-k_m^2-k_n^2}{2k_m k_n}\right] C_{lmn}\eta_l\dot{\eta}_m + \delta_{1n} C_1 k \eta_1^2 \dot{\eta}_1, \tag{A 3a}$$

$$C_1 = -\tfrac{5}{6}\langle\psi_1^4\rangle + \tfrac{1}{2}C_{11n}^2[(k_n/k)^2+\tfrac{1}{2}(k_n/k)^3], \tag{A 3b}$$

in the present approximation.

Substituting (A 2) into (A 1) and simplifying the result with the aid of the Helmholtz equation (2.2) for ψ_n yields the quartic truncation

$$F = 2\nu\phi_m\phi_n\langle[k_m^2k_n^2\psi_m\psi_n + k_m k_n \nabla\psi_m\cdot\nabla\psi_n + \psi_{mxy}\psi_{nxy} - \psi_{mxx}\psi_{nyy}] \times [(k_m+k_n)^{-1} + \eta_l\psi_l + \tfrac{1}{2}(k_m+k_n)\eta_j\eta_l\psi_j\psi_l]\rangle, \tag{A 4}$$

† This condition excludes water except under rather special circumstances, but it appears to be a good approximation for the n-butyl alcohol used by Tufillaro *et al.* (1989).

which, through (A 3) and (2.4) and (2.5) for the ψ_n, reduces to

$$F = 2\nu k\{\dot{\eta}_1^2[1+\tfrac{1}{2}k\eta_2+k\eta_4+(\tfrac{11}{4}+\sqrt{2})\,k^2\eta_1^2] - k\eta_1\,\dot{\eta}_1[(\tfrac{3}{2}\sqrt{2}-2)\,\dot{\eta}_2+\dot{\eta}_4]+\sqrt{2}\dot{\eta}_2^2+2\dot{\eta}_4^2\}, \tag{A 5}$$

for the square pattern. Substituting η_n from (3.3a) and (3.7) into (A 5), averaging over ωt, and comparing the result with (4.6b), we obtain

$$\delta = 2\nu k^2/\omega \tag{A 6a}$$

and

$$\Gamma/\delta = \tfrac{11}{4}+\sqrt{2}+\tfrac{1}{2}\kappa_2^{-1}+\kappa_4^{-1}+\tfrac{1}{4}(32\sqrt{2}-45)\,\Omega_2^{-1}-\tfrac{3}{2}\Omega_4^{-1}+(17\sqrt{2}-24)\,\Omega_2^{-2}+2\Omega_4^{-2}. \tag{A 6b}$$

Invoking $kd \gg 1$, reversing the sign of σ in (3.13) to obtain $\hat{C}$, and combining the result with (A 6b) in (4.10b), we obtain

$$\gamma/\delta = 3+\tfrac{1}{2}\sqrt{2}+\tfrac{15}{16}\sigma+\tfrac{1}{2}\kappa_4^{-1}+(5\sqrt{2}-7)\,\Omega_2^{-1}-\tfrac{5}{4}\Omega_4^{-1}+(17\sqrt{2}-24)\,\Omega_2^{-2}+2\Omega_4^{-2}. \tag{A 7}$$

The corresponding approximation to P, (3.14), is

$$P/\epsilon = -(3-2\sqrt{2})\,(\kappa_2\,\Omega_2)^{-1}+(\kappa_4\,\Omega_4)^{-1}. \tag{A 8}$$

Appendix B. Rolls

The normal modes for one-dimensional Faraday waves, determined as in §2, are

$$\psi_1 = \sqrt{2}\cos kx \quad (k_1 = k), \qquad \psi_2 = \sqrt{2}\cos 2kx \quad (k_2 = 2k). \tag{B 1a,b}$$

Proceeding as in §3, we obtain the average Lagrangian in the form (3.10) and (3.11) with

$$C = \tfrac{1}{2}+\tfrac{1}{4}(1+T^2)^2\,\kappa^{-1}-\tfrac{1}{8}(3-T^2)^2\,\Omega^{-1}+\tfrac{9}{8}T^2\sigma \tag{B 2}$$

and

$$P = \tfrac{1}{2}\epsilon(3-T^2)\,(1+T^2)\,(\kappa\Omega)^{-1}, \tag{B 3}$$

where

$$\kappa \equiv \frac{1+4\ell^2}{1+\ell^2}, \quad \Omega \equiv 1+T^2-\kappa, \quad \sigma \equiv \frac{\ell^2}{1+\ell^2}, \quad T \equiv \tanh kd. \tag{B 4a–d}$$

The corresponding dissipation function, calculated through (A 3) and (A 4), is given by (for $kd \gg 1$)

$$F = 2\nu k[\dot{\eta}_1^2(1-\tfrac{1}{2}k^2\eta_1^2)+\sqrt{2}k\dot{\eta}_1(\dot{\eta}_1\,\eta_2-\eta_1\,\dot{\eta}_2)+2\dot{\eta}_2^2], \tag{B 5}$$

which leads to (A 6a) for δ,

$$\Gamma/\delta = -\tfrac{1}{2}+2\kappa^{-1}-3\Omega^{-1}+4\Omega^{-2}, \tag{B 6}$$

and

$$\gamma/\delta = -1+\kappa^{-1}-\tfrac{5}{2}\Omega^{-1}+4\Omega^{-2}+\tfrac{9}{8}\sigma. \tag{B 7}$$

Appendix C. Comparison with Ezerskii *et al.* (1986)

Letting $a_+ = a_- = b_+ = b_- = a$ in E(2) (E(2) stands for Ezerskii *et al.* equation (2)), we obtain

$$\zeta = 2\,\mathrm{Re}\,(a\,\mathrm{e}^{-\mathrm{i}\omega t})\,(\cos kx+\cos ky), \tag{C 1}$$

which may be equated to $\eta_1\,\psi_1$ in (2.1), with η_1 and ψ_1 given by (3.3a) and (2.4), to obtain

$$a_+ = a_- = b_+ = b_- = a = \epsilon^{\frac{1}{2}}k^{-1}(p+\mathrm{i}q) \tag{C 2}$$

in the present notation. Substituting (C 2) into either E($3a$) or E($3b$), setting $\gamma = \delta\omega$ therein, neglecting spatial modulation except insofar as it implies the detuning $\omega - \omega_k$, and introducing $\tau = \epsilon\omega t$ ($3.3b$), we obtain

$$(\partial_\tau + \alpha)(p + \mathrm{i}q) = \mathrm{i}(p - \mathrm{i}q) + \mathrm{i}[\beta + C(p^2 + q^2)](p + \mathrm{i}q), \tag{C 3}$$

where $\alpha \equiv \delta/\epsilon$, $\beta \equiv (\omega - \omega_k)/(\epsilon\omega)$ (cf. (3.12)),

$$C = (F + 2R + S + T)(\omega k^2)^{-1} = 1.38, \tag{C 4}$$

and F, R, S, T are the coefficients given just below E($3b$). (C 3) is equivalent to (5.1) except for the numerical value of C (1.38 *vs.* 1.89 in the limit $kd, kl_* \uparrow \infty$) and the neglect of third-order damping and parametric forcing.

Appendix D. Comparison with Milner (1991)

Milner assumes $kd, kl_* \gg 1$ and expands the free-surface displacement for the primary mode in plane waves according to (in the present notation with spatial modulation suppressed)

$$\eta = a_j(\tau)\exp(\mathrm{i}\boldsymbol{k}_j \cdot \boldsymbol{x} - \mathrm{i}\omega t) + \text{c.c.}, \tag{D 1}$$

where the summation is over a set of unit wave vectors and their opposites, $\boldsymbol{k}_{-j} \equiv \boldsymbol{k}_j$, that form an equiangular star for $j = 1, \ldots, N$, and c.c. is the complex-conjugate of that sum. A regular pattern is obtained by choosing equal amplitudes, $a_j = a$, with $N = 1$, 2 or 3, respectively, for rolls, squares or hexagons. Equating the spatial mean squares of (D 1) and the primary component of (2.1) and invoking $kd \gg 1$ (so $l = 2\epsilon^{\frac{1}{2}}k^{-1}$) and (3.3), we obtain

$$a = (8N)^{-\frac{1}{2}} l(p + \mathrm{i}q) = (\epsilon/2N)^{\frac{1}{2}} k^{-1}(p + \mathrm{i}q). \tag{D 2}$$

Milner's evolution equation (22), after letting $a_j = a$, $f = -4a_0\omega^2$ ($f\cos 2\omega t$ is the imposed acceleration) and $\gamma^{(0)} = \delta\omega$, and suppressing the spatial modulation, reduces to

$$(\epsilon\partial_\tau + \delta)a - \mathrm{i}\epsilon a^* + (\tilde{\Gamma} - \mathrm{i}\tilde{T})k^2|a|^2 a = 0, \tag{D 3}$$

wherein a^* is the complex-conjugate of a and

$$\tilde{\Gamma} \equiv \sum_l \left[\frac{\gamma_{jl}^{(1)} + \gamma_{jl}^{(2)}}{k^2\omega}\right] \equiv \frac{\Gamma}{k^2\omega}, \quad \tilde{T} \equiv \sum_l \left[\frac{T_{jl}^{(1)} + T_{jl}^{(2)}}{k^2\omega}\right] \equiv \frac{T}{k^2\omega} \tag{D 4a, b}$$

(which are independent of j by virtue of symmetry). Substituting a from (D 2) into (D 3) and comparing the result with (5.1) after setting $\beta = P = 0$ therein (since Milner neglects resonant offset and parametric excitation of the secondary modes in his (22)), we obtain

$$(\gamma, C) = (2N)^{-1}(\tilde{\Gamma}, \tilde{T}). \tag{D 5}$$

Milner's (A 1, 2) yield $\tilde{T} = 23/2$ and $\tilde{T} = \frac{1}{2}(25 - \surd 2) = 11.79$ for $N = 1$ and 2, respectively, which compare with $2C = 17/4$ and $4C = \frac{1}{4}(33 - 2\surd 2) = 7.54$ from the limiting ($k \uparrow \infty$) values of (B 2) and (3.13). He gives (p. 89) $\Gamma/\nu k^4 = (6, 1.93)$ for $N = (1, 2)$, which (since $\delta = 2\nu k^2/\omega$) imply $\tilde{\Gamma}/\delta = (3, 0.97)$ and $\gamma/\delta = (\frac{3}{2}, 0.24)$ and compare with $\gamma/\delta = (21/8, 6.04)$ from the limiting values of (B 6), (A 7). The corresponding values of P/δ are $(-\frac{1}{4}, -0.23)$.

Milner gives few details of his derivations, and I have been unable to determine just how and where our analytical results differ. However, I do not agree with his equation (20) for 'the Hamiltonian implied by the energy equation,' which he implicitly equates to the mean energy $\langle E\rangle$ (in the present notation). There appears to be a dimensional error in his definition '$h^{(i)}_{jk} = -\Sigma k^4 T^{(i)}_{jk}$' ($\Sigma$ is the surface tension), but if $T^{(i)}_{jk}$ is replaced by $T^{(i)}_{jk}/(k^2\omega)$, the assumption $a_j = a$, the approximation $\omega^2 \approx \Sigma k^3$, and (D 4$b$) reduce Milner's (20) to

$$\langle E\rangle = 2N[(2\omega^2/k)|a|^2 - \tilde{T}\omega^2 k|a|^4] \tag{D 6a}$$

$$= \tfrac{1}{2}a_0\,\omega^2 l^2[\epsilon^{-1}(p^2+q^2) - (\tilde{T}/4N)(p^2+q^2)^2], \tag{D 6b}$$

where (D 6b) follows from (D 6a) through (D 2) and $\epsilon \equiv ka_0$. This result should be equal to (4.5b); in fact, the quadratic terms are equal (for $\omega = \omega_1$), but the coefficients of $a_0\,\omega^2 l^2(p^2+q^2)^2$ are $-(\hat{T}/8N) = -\frac{1}{4}C$ (from (D 5)) and $\frac{1}{4}\hat{C}$, respectively.

Appendix E. Liapunov functionals

A Liapunov functional $\mathscr{L}(A)$ for the evolution equation

$$\dot{A} = F(A) \tag{E 1}$$

in some neighbourhood of a fixed point $A = A_*$ must satisfy (Manneville 1990, pp. 29ff)

$$\mathscr{L}(A_*) = 0, \quad \mathscr{L}(A \neq A_*) > 0, \quad \mathrm{d}\mathscr{L}/\mathrm{d}t \leqslant 0. \tag{E 2a–c}$$

The functional

$$\mathscr{L}_*(A) = -\int_{A_*}^{A} F(A)\,\mathrm{d}A \tag{E 3}$$

manifestly satisfies (E 2a, c), but whether it satisfies (E 2b) depends on both $F(A)$ and A_*. It is worth emphasizing that $\mathscr{L}_*$ is not unique; in particular, the transformation $A = \hat{A}^\lambda$, $\lambda > 0$, yields a one-parameter family $\mathscr{L}_*(\hat{A};\lambda)$.

Milner's functional for (6.6) is (in his approximation but in the present notation)

$$V \equiv \mathscr{L}_0(A) = \tfrac{1}{2}A^2[-(1-B^2) + \tfrac{1}{2}(1+\mu B)A^2 + O(\mu^2 A^4)]. \tag{E 4}$$

The corresponding approximation to the Faraday-wave fixed point is given by

$$A^2 = \mathscr{A}_+ = (1+\mu B)^{-1}(1-B^2) + O(\mu^2), \tag{E 5}$$

and $\mathscr{L}_0 > 0$ in $B^2 < 1$ (the domain of interest for Faraday waves) if and only if $A^2 > 2\mathscr{A}_+$. It follows that V is not a proper Liapunov functional for the Faraday-wave fixed point.

A Liapunov functional for the evolution equation (5.7) and the fixed point $\mathscr{A}_+$ is given by

$$\mathscr{L}_+(\mathscr{A}) = -(1-B^2)(\mathscr{A}^2 - \mathscr{A}_+^2) + \tfrac{2}{3}(1+\mu B)(\mathscr{A}^3 - \mathscr{A}_+^3) + \tfrac{1}{8}\mu^2(\mathscr{A}^4 - \mathscr{A}_+^4) \tag{E 6a}$$

$$= \tfrac{1}{2}\mu^2(\mathscr{A} - \mathscr{A}_+)^2[\tfrac{1}{2}\mathscr{A}_+(\mathscr{A}_+ - \mathscr{A}_-) + \tfrac{1}{3}(2\mathscr{A}_+ - \mathscr{A}_-)(\mathscr{A} - \mathscr{A}_+) + \tfrac{1}{4}(\mathscr{A} - \mathscr{A}_+)^2], \tag{E 6b}$$

where $\mathscr{A}_\pm$ are given by (6.1). It can be shown that $0 < \mu < 1$ and $B^2 < 1$ are sufficient conditions for $\mathscr{L}_+(\mathscr{A}) > 0$, and, hence for the asymptotic stability of $\mathscr{A} = \mathscr{A}_+$ (subject to the *a priori* neglect of spatial modulation), as also may be inferred from a linear stability analysis of (5.7).

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